THE ISING MODEL ON CUBIC MAPS: ARBITRARY GENUS (PRELIMINARY VERSION)

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Abstract. We obtain a recursive algorithm to compute the partition function of cubic Ising maps with fixed size and genus. The algorithm runs in polynomial time, which is much faster than Tutte-like/topological recursion methods.

We construct this algorithm out of a partial differential equation that we derive from the first equation of the KP hierarchy for the generating function of bipartite maps, which is related to the Ising partition function by a change of variables. We also obtain inequalities on the coefficients of this partition function, that will be useful for a probabilistic study of cubic Ising maps whose genus grows linearly with their size.

1. Introduction

Combinatorial maps are a model of discrete surfaces formed by gluing polygons along their sides, or alternatively, of graphs embedded on surfaces. Their study goes back to the 60s, where Tutte performed the enumeration of *planar maps* (i.e. maps of the sphere) in a series of papers [24, 23, 25]. Later on, these results were extended to maps on a surface of fixed genus, both as exact algorithms to compute the numbers of maps recursively in the genus, and as asymptotics as the size of the maps (i.e. the number of edges) grows large [26]. These algorithms were later shown to fit in a larger framework of enumerative geometry, namely the *topological recursion* (see [12] and references therein). However the TR algorithms have a superexponential runtime in the genus, and they cannot be used in practice to compute numbers except when the genus is very small.

Fortunately, there are faster ways of computing these numbers: the generating series of maps satisfies the *KP hierarchy*, an integrable family of nonlinear partial differential equation (hereafter PDEs) – see [19] and references therein. These PDEs arose in the context of mathematical physics (as an extension of the more classical *KdV hierarchy*, which models waves in shallow water, but was already shown to appear in enumerative geometry [27, 16]). Goulden and Jackson proved this fact [13], and derived a fast (polynomial time) algorithm in the form of a simple recursion for the numbers of triangulations (maps whose faces are triangles) in any genus (it turned out that this recursion had already been observed in the physics litterature [10, 5, 1]). This happened in parallel with several works on Hurwitz numbers ([21, 11]), and later on, several other fast recursions for combinatorial maps were derived [7, 18].

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In the language of physics, "usual" maps correspond to a model of two dimensional quantum gravity "without matter", but one can also consider maps endowed with a model of statistical mechanics (see [9] for a detailed review). In this paper, we consider the particular case of the *Ising model*, which consists of putting colors (or spins) on the faces of our maps, and introducing a weight on monochromatic edges. The Ising model was first introduced on fixed lattices [17, 14], and has been an active field of study ever since. The Ising model on random maps as a model of "2D quantum gravity with matter" was introduced more recently [3]. As for usual maps, the Ising model has been solved exactly in the planar case [15, 4], and it was shown to satisfy the topological recursion, see e.g. [12]. (Note that all these results rely on a correspondence between Ising maps and bipartite maps, that we also exploit in the present paper.) However, once again the algorithmic runtime grows exponentially with the genus. Our main result is to give a fast algorithm for the associated counting problem.

Theorem 1.1. *Let $I_{n,g}$ be the partition function of labelled triangulations with $2n$ faces colored either black or white faces, with a weight $\nu_\circ$ (resp. $\nu_\bullet$) per white-white (resp. black-black) edge. The series $I = \sum_{n,g\geq 0} t^{3n} s^g I_{n,g}$ satisfies and explicit PDE in the variables t, $\nu_\circ$ and $\nu_\bullet$, see (19) for details. This equation yields an algorithm to compute the polynomials $I_{n,g}$ in polynomial time (and space) with respect to n and g.*

Note that since we have two different Ising weights $\nu_\circ$ and $\nu_\bullet$, this is equivalent to studying the Ising model coupled with an external magnetic field. The Ising model without magnetic field corresponds to setting $\nu_\circ = \nu_\bullet$.

A corollary of our main theorem is the following enumerative bound.

Theorem 1.2. *Let $T_{n,g}(\nu_\bullet, \nu_\circ)$ be the the partition function of rooted triangulations with $2n$ faces colored either black or white faces, with a weight $\nu_\circ$ (resp. $\nu_\bullet$) per white-white (resp. black-black) edge. For $\nu_\bullet, \nu_\circ > 0$ the following inequality holds:*

$$nT_{n,g}(\nu_\bullet,\nu_\circ) \geq C(\nu_\bullet,\nu_\circ)\left(n^3 T_{n-2,g-1}(\nu_\bullet,\nu_\circ) + \sum_{\substack{n_1+n_2=n-2\\ g_1+g_2=g}} n_1 T_{n_1,g_1}(\nu_\bullet,\nu_\circ) n_2 T_{n_1,g_2}(\nu_\bullet,\nu_\circ)\right) \tag{1}$$

where

$$C(\nu_\bullet,\nu_\circ) = 10\frac{\min(\nu_\circ^2,\nu_\bullet)^4}{\nu_\circ^2+\nu_\bullet+\frac{\nu_\circ}{\nu_\bullet}+\frac{1}{\nu_\circ}}.$$

Similar bounds for usual triangulations can be derived from the Goulden–Jackson recurrence formula [13], and they are a crucial ingredient in a work of Budzinski and the third author [6] that establishes the local convergence of random high genus triangulations. The bounds we provide will play the same role for the study of high genus Ising triangulations.

Our method consists in using the Goulden–Jackson result, coupled with the fact that the Ising generating series is the same as the bipartite map generating series up to a change of variables. We then express the first KP equation in the relevant variables, and although it does not give a closed recurrence formula like it did for usual triangulations, the PDE that we obtain can still be used to calculate the polynomials $I_{n,g}$ recursively.

The structure of the paper is as follows. In Section 2, we recall the first equation of the KP hierarchy satisfied by the generating function of labelled biparite maps, and show how they specialise to biparite maps having only vertices of degree 2 and 3. In Section 3, we state a relation between these maps and cubic maps endowed with the Ising model, that induces a new PDE for the generating function of the latter, as announced in Theorem 1.1. In Section 4, we show that this PDE, along with minimal assumptions, characterises the generating function of Ising cubic maps, and yields the desired polynomial-time algorithm to compute its coefficients. In Section 5, we consider the specialisation of this PDE in three cases: planar and unicellular Ising maps, as well as uncoloured cubic maps (which recovers the Goulden–Jackson recursion on triangulations). Finally, in Section 6 we prove Theorem 1.2.

2. Bipartite maps with vertex degrees 2 and 3

2.1. Definitions

We first consider in this section edge-labelled bipartite maps of arbitrary genus, in which all vertices have degree 1, 2 or 3. We call such maps, for short, *maps of bounded degree*. We will count families of such maps by edges (variable z), faces (variable u), white vertices of degree $i \in \{1,2,3\}$ (variable p_i), and black vertices of degree i (variable q_i). We sometimes call these variables the *weights* of the corresponding items (edges, faces, etc.). Our generating functions are exponential in the number of edges. In particular, we denote by H the generating function of bipartite maps (also called *hypermaps*, hence the notation) of bounded degree . That is,

$$H = \sum_{\mathfrak{m}} \frac{z^{\mathrm{e}(\mathfrak{m})}}{\mathrm{e}(\mathfrak{m})!} u^{\mathrm{f}(\mathfrak{m})} \prod_{i=1}^{3} p_i^{v_i^\circ(\mathfrak{m})} q_i^{v_i^\bullet(\mathfrak{m})}, \tag{2}$$

where the functions e, f, v_i° and $v_i^\bullet$ count respectively edges, faces, white vertices of degree i and black vertices of degree i. The sum runs over all labelled bipartite maps $\mathfrak{m}$ satisfying the above small degree condition. We call *leaves* the vertices of degree 1. Note that the variable z is redundant, since

$$\mathrm{e}(\mathfrak{m}) = \sum_i i v_i^\circ(\mathfrak{m}) = \sum_i i v_i^\bullet(\mathfrak{m}).$$

Also, the genus $\mathrm{g}(\mathfrak{m})$ of the map $\mathfrak{m}$ is given by Euler's relation:

$$\sum_i \left(v_i^\circ(\mathfrak{m}) + v_i^\bullet(\mathfrak{m})\right) + \mathrm{f}(\mathfrak{m}) - \mathrm{e}(\mathfrak{m}) = 2 - 2\,\mathrm{g}(\mathfrak{m}).$$

This series starts as follows:

$$H = p_1 q_1 z u + \left(p_1^2 q_2 u + p_2 q_1^2 u + p_2 q_2 u^2\right)\frac{z^2}{2}$$
$$+ \left(p_1^3 q_3 u + p_3 q_1^3 u + 3p_1 p_2 q_1 q_2 u + 3p_1 p_2 q_3 u^2 + 3p_3 q_1 q_2 u^2 + p_3 q_3 u^3 + p_3 q_3 u\right)\frac{z^3}{3} + \mathcal{O}\left(z^4\right).$$

See Figure 1 for an illustration of the coefficient of $z^3/3!$. These labelled maps, say with n edges, can be encoded by three permutations of the symmetric group $\mathfrak{S}_n$, denoted $\sigma_\circ$, $\sigma_\bullet$ and ϕ, that describe the cyclic order of edge labels around white vertices, black vertices, and faces, respectively. The product $\sigma_\circ \sigma_\bullet \phi$ is the identity, and the group generated by these three permutations acts transitively on $\{1,\ldots,n\}$.

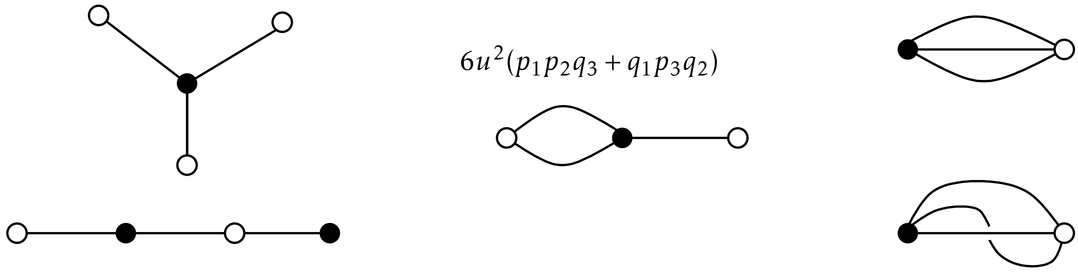

Figure 1. Bipartite maps with 3 edges and vertex degrees at most 3. The weights keep track of the number of labellings of the edges and of the exchange of colours.

2.2. The first KP equation

In 2008, Goulden and Jackson proved that the series H satisfies the partial differential equations of the KP hierarchy [13, Thm. 3.1]. This is actually true as well without the bound of the degrees. Here we shall use the first of these partial differential equations (PDE) only. It involves derivatives with respect to the three variables p_i.

Theorem 2.1. *The above series H satisfies the following fourth order partial differential equation:*

$$H_{1,3} = H_{2,2} + \frac{1}{12}H_{1,1,1,1} + \frac{1}{2}(H_{1,1})^2, \tag{3}$$

where an index i indicates a partial derivative in the variable p_i.

Our objective in this section is to derive a PDE for the specialization of H at $p_1 = q_1 = 0$. That is, we want an equation for bipartite maps having vertices of degree 2 and 3 only. For such a map $\mathfrak{m}$, we have

$$\mathrm{e}(\mathfrak{m}) = 2v_2^\circ(\mathfrak{m}) + 3v_3^\circ(\mathfrak{m}) = 2v_2^\bullet(\mathfrak{m}) + 3v_3^\bullet(\mathfrak{m}), \tag{4}$$

hence the variables p_3 and q_3 are redundant if we keep z, p_2 and q_2. Later we will set them to 1, but for the moment we keep them. Let Θ be the operator that sets the variables p_1 and q_1 to 0, and let $B := \Theta H$. That is,

$$B = \sum_{\mathfrak{m}} \frac{z^{\mathrm{e}(\mathfrak{m})}}{\mathrm{e}(\mathfrak{m})!} u^{\mathrm{f}(\mathfrak{m})} \prod_{i=2}^{3} p_i^{v_i^\circ(\mathfrak{m})} q_i^{v_i^\bullet(\mathfrak{m})}, \tag{5}$$

where the sum runs over edge labelled bipartite maps with degrees 2 and 3. The set of such maps will be denoted by $\mathcal{B}$.

The following proposition, based on combinatorial constructions, will allow us to specialize (3) at $p_1 = q_1 = 0$ (see Corollary 2.3).

Proposition 2.2. *Let us introduce the following linear differential operator:*

$$L = \frac{2}{1 - z^2p_2q_2}\left(z^2q_2p_3\frac{\partial}{\partial p_2} + zq_3\frac{\partial}{\partial q_2}\right). \tag{6}$$

Then the partial derivatives occurring in Theorem 2.1, once specialized at $p_1 = q_1 = 0$, are given by

$$\Theta H_{1,1} = L^2 B + \frac{uz^2 q_2}{1 - z^2 p_2 q_2}, \tag{7}$$

$$\Theta H_{1,1,1,1} = L^4 B + \frac{12 z^6 u \left(p_3^2 q_2^5 z^4 + 2 p_3 q_2^2 q_3 z + p_2 q_3^2\right)}{(1 - z^2 p_2 q_2)^5}, \tag{8}$$

$$\Theta H_{1,3} = \frac{1}{3p_3}\left(z\frac{\partial}{\partial z} - 2p_2 \frac{\partial}{\partial p_2} - 1\right) LB, \tag{9}$$

and finally

$$\Theta H_{2,2} = \frac{\partial^2 B}{\partial p_2^2}. \tag{10}$$

Proof. The enumeration of labelled objects, and the correspondence between operations on these objects and operations on their generating functions, is conveniently described using the notion of *species*. We refer to the book by Bergeron, Labelle and Leroux [2], or to the short, self-contained account of [22, Sec. 1]. The two ingredients that we need here are the product of two species, in which the labels are distributed over the two objects, and the marking of an unlabelled element, here a vertex.

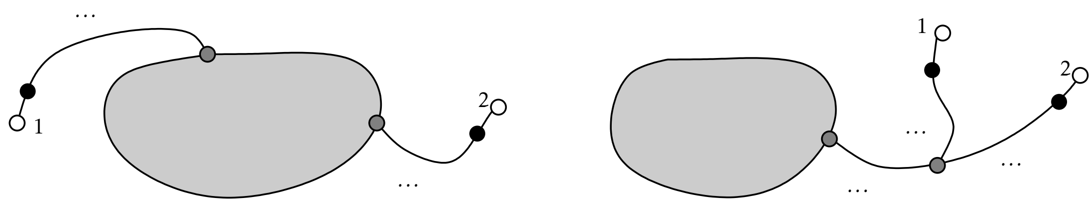

FIGURE 2. Bipartite maps with two ordered white leaves and at least one vertex of degree 3.

Let us begin with $\Theta H_{1,1}$. This series counts maps of bounded degree having exactly two (ordered, unweighted) leaves, both white. There are two types of such maps:

- Maps with no vertex of degree 3. They are reduced to a chain of alternatingly black and white vertices, with a white leaf at both ends, oriented from the first to the second leaf. In particular, the chain contains an even number, say $2n$, of edges. A chain of length $2n$ is then simply encoded by a permutation of $\mathfrak{S}_{2n}$ (giving the sequence of edge labels from end to end), and the contribution of maps of this type is
$$u \sum_{n \geq 1} \frac{z^{2n}}{(2n)!}(2n)! p_2^{n-1} q_2^n = \frac{uz^2 q_2}{1 - z^2 p_2 q_2}.$$
- Maps with at least one vertex of degree 3: in these maps, each of the two leaves lies at the end of a chain of vertices of degree 2, attached at a vertex of degree 3 (Figure 2). In this case, let us first delete the chain of the second leaf: we obtain a new map with small degrees and one white leaf. We then repeat this operation with the first leaf so as to end with a map of degrees 2 and 3. Conversely, given a set $\mathcal{M}$ of labelled bipartite maps with small degrees, let $\overrightarrow{\mathcal{M}}$ be the set of maps obtained as follows: take a map in $\mathcal{M}$, choose a vertex v of degree 2 and one of the two corners at this

vertex, and attach at this corner a chain ending with a white leaf; the vertex v has now degree 3. If M denotes the generating function of $\mathcal{M}$ (with variables z, u, p_i, q_i as in (2)), the generating function of $\overrightarrow{\mathcal{M}}$ is

$$\overrightarrow{M} := 2p_1p_3\frac{z^2q_2}{1-z^2p_2q_2}\frac{\partial M}{\partial p_2} + 2p_1q_3\frac{z}{1-z^2p_2q_2}\frac{\partial M}{\partial q_2} = p_1LM,$$

where L is defined by (6). The first (resp. second) half of the expression counts maps such that the chain is attached to a white (resp. black) vertex. In this case, the chain has to be of even (resp. odd) length.

Now the maps that we need to count are precisely those obtained by first adding to a map of $\mathcal{B}$ a chain, ending at a first white leaf, and then a second chain to the resulting map, ending at a second white leaf. The above argument gives the resulting generating function as L^2B. This concludes the proof of (7).

The series $\Theta H_{1,1,1,1}$ counts maps with small labels, having exactly four (ordered, unweighted) leaves, all of them being white. These maps are obtained by adding consecutively two chains ending at white leaves to maps counted by $\Theta H_{1,1}$. By the above construction, $\Theta H_{1,1,1,1} = L^2\Theta H_{1,1}$, and thanks to (7), this yields the announced result (8).

The series $\Theta H_{1,3}$ counts maps with small labels, having exactly one (unweighted) leaf which is white, and in addition a marked white unweighted vertex of degree 3. These maps are obtained from maps of $\mathcal{B}$ by first adding a chain ending at a white leaf — this gives the series LB — and then marking a white vertex of degree 3. The resulting generating function is

$$\frac{\partial}{\partial p_3}LB.$$

The maps $\mathfrak{m}$ counted by LB satisfy

$$\mathrm{e}(\mathfrak{m}) = 1 + 2v_2^\circ(\mathfrak{m}) + 3v_3^\circ(\mathfrak{m}).$$

Hence

$$\frac{\partial}{\partial p_3}LB = \frac{1}{3p_3}\left(z\frac{\partial}{\partial z} - 2p_2\frac{\partial}{\partial p_2} - 1\right)LB,$$

which gives (9).

The final identity is obvious, since the specialization operator Θ commutes with the differentiation with respect to p_2. □

We can now convert the KP equation (3) into an equation for bipartite maps with vertices of degree 2 and 3.

Corollary 2.3. *The generating function B of bipartite maps having only vertices of degree 2 and 3 satisfies the following fourth order partial differential equation:*

$$\frac{1}{3p_3}\left(z\frac{\partial}{\partial z} - 2p_2\frac{\partial}{\partial p_2} - 1\right)LB = \frac{\partial^2 B}{\partial p_2^2} + \frac{1}{12}L^4B + \frac{1}{2}\left(L^2B\right)^2 + \frac{uz^2q_2}{1-z^2p_2q_2}L^2B + R, \qquad (11)$$

where L is defined by (6) and

$$R := \frac{1}{2}\left(\frac{uz^2q_2}{1-z^2p_2q_2}\right)^2 + \frac{z^6u\left(p_3^2q_2^5z^4 + 2p_3q_2^2q_3z + p_2q_3^2\right)}{(1-z^2p_2q_2)^5}.$$

This is again a PDE in three variables, namely z, p_2 and q_2.

Remark 2.4. The above PDE does not characterize the series B. Experimentally, if we prescribe the following form for B:

$$B = \sum_{i,j,n} B_{i,j,n} p_2^i q_2^j z^n,$$

where the $B_{i,j,n}$ are polynomials in u such that $B_{i,j,n} = B_{j,i,n}$, and the summation is restricted to tuples $(i,j,n) \in \mathbb{N}^3$ subject to the natural conditions

$$n \geq 2, \quad 0 \leq i \leq n/2, \quad 0 \leq j \leq n/2, \quad i \equiv -n \mod 2, \quad \text{and} \quad j \equiv -n \mod 2,$$

then the solution of the PDE appears to be unique if we prescribe in addition the values $B_{n,0,0}$ of the polynomials that count (by faces) bipartite cubic maps with $3n$ edges. See our Maple session for details. This should be provable along the same lines as Proposition 4.1 below.

Remark 2.5. The above PDE is not symmetric in p_2 and q_2, while the series B *is* symmetric. One way to obtain an *antisymmetric* PDE for B, which in addition does not involve derivatives in z anymore, is the following. The terms of the PDE that contain z-derivatives are $B_{z,p_2} := \partial^2 B/\partial z \partial p_2$ and $B_{z,q_2} := \partial^2 B/\partial z \partial q_2$, and the PDE depends linearly on them. The symmetric PDE contains these two terms as well. So we write both PDEs, solve them for these two derivatives, and finally write $\partial_{q_2} B_{z,p_2} = \partial_{p_2} B_{z,q_2}$. This gives a PDE in p_2 and q_2 only, but of order 5 instead of 4. Of course, an alternative is to take any (symmetric) combinattion of the original PDE and its symmetric.

3. Ising cubic maps

In this section we consider the class of *cubic* maps (all vertices have degree 3), labelled on *half-edges*. We equip them with an Ising model, meaning that their vertices are colored black and white as in Section 2, but now adjacent vertices may get the same color. In this case we say that the edges that join them are *monochromatic* (or, for short, white, or black). Observe that for any cubic map $\mathfrak{m}$, we have $2\,\mathrm{e}(\mathfrak{m}) = 3\,\mathrm{v}(\mathfrak{m})$, so that the number of edges (resp. vertices) is a multiple of 3 (resp. 2).

We will count these coloured cubic maps — called *Ising maps* henceforth — by the number of edges (variable t), the genus (variable s), the number e° (resp. $\mathrm{e}^\bullet$) of monochromatic white (resp. black) edges (variables $\nu_\circ$ and $\nu_\bullet$, respectively). Note that a power t^{3n} corresponds to a map with $3n$ edges and $2n$ vertices. Let I be the exponential generating function of Ising maps, labelled on half-edges:

$$I = \sum_{\mathfrak{m}} \frac{t^{\mathrm{e}(\mathfrak{m})}}{(2\,\mathrm{e}(\mathfrak{m}))!} s^{\mathrm{g}(\mathfrak{m})} \nu_\circ^{\mathrm{e}^\circ(\mathfrak{m})} \nu_\bullet^{\mathrm{e}^\bullet(\mathfrak{m})} = \left(\frac{1}{3}(1+s) + \nu_\circ \nu_\bullet + (\nu_\bullet^3 + \nu_\circ^3)(\frac{2}{3} + \frac{s}{6})\right) t^3 + \mathcal{O}(t^6). \tag{12}$$

See Figure 3 for a justification of the coefficient of $t^3/6!$. Observe that the number of bicoloured (or: *frustrated*) edges is $\mathrm{e}^{\bullet\circ}(\mathfrak{m}) = \mathrm{e}(\mathfrak{m}) - \mathrm{e}^\bullet(\mathfrak{m}) - \mathrm{e}^\circ(\mathfrak{m})$, and that the numbers of black and white vertices are given by

$$3\,\mathrm{v}^\bullet(\mathfrak{m}) = \mathrm{e}^{\bullet\circ}(\mathfrak{m}) + 2\,\mathrm{e}^\bullet(\mathfrak{m}), \qquad \text{and} \qquad 3\,\mathrm{v}^\circ(\mathfrak{m}) = \mathrm{e}^{\bullet\circ}(\mathfrak{m}) + 2\,\mathrm{e}^\circ(\mathfrak{m}),$$

so that these numbers are recorded (be it implicitely) in our series. This means that the Ising model that we address includes a magnetic field. Also, the above identities imply that $\mathrm{e}^\bullet(\mathfrak{m}) - \mathrm{e}^\circ(\mathfrak{m})$ is always 0 modulo 3.

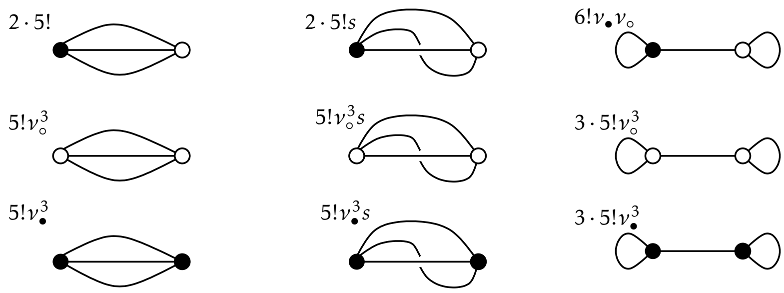

FIGURE 3. Ising cubic maps with 3 edges and 2 vertices. The weights keep track of the number of labellings of half-edges.

Starting from an Ising cubic map, one can insert on its edges bicoloured chains of vertices of degree 2 so as to obtain a bipartite map with vertex degrees 2 and 3. This allows one to relate the series I to the series B of Section 2. Variants of this "trick" have been used several times in the study of the Ising model on maps [15, 4, 12]. We work out its details, in our setting, in Section 3.1. Then we use this in Section 3.2 to convert the PDE satisfied by B into a PDE satisfied by I.

3.1. FROM BIPARTITE MAPS TO ISING MAPS

Proposition 3.1. *Define the change of variables Ψ on the ring $\mathbb{Q}[\nu_\bullet, \nu_\circ, s][[t]]$ by*

$$\begin{aligned} t &\mapsto u^{1/3} \frac{z}{1 - z^2 p_2 q_2}, & \nu_\bullet &\mapsto z p_2, \\ s &\mapsto u^{-2}, & \nu_\circ &\mapsto z q_2. \end{aligned} \tag{13}$$

It gives a series of $\mathbb{Q}[p_2, q_2, u^{1/3}, 1/u][[z]]$. Then the generating function B of bipartite maps with vertex degrees 2 and 3, defined by (5) and evaluated at $p_3 = q_3 = 1$, is

$$B = \frac{u^2}{2} \log\left(\frac{1}{1 - z^2 p_2 q_2}\right) + u^2 \Psi(I), \tag{14}$$

where I is the Ising generating function defined by (12). Conversely,

$$I = s\Phi(B) - \frac{1}{2} \log \frac{1}{1 - \nu_\bullet \nu_\circ}, \tag{15}$$

where the inverse change of variables Φ is defined by

$$\begin{aligned} z &\mapsto s^{1/6} t(1 - \nu_\bullet \nu_\circ), & p_2 &\mapsto \frac{\nu_\bullet}{s^{1/6} t(1 - \nu_\bullet \nu_\circ)}, \\ u &\mapsto s^{-1/2}, & q_2 &\mapsto \frac{\nu_\circ}{s^{1/6} t(1 - \nu_\bullet \nu_\circ)}. \end{aligned} \tag{16}$$

The transformation Φ maps series in $\mathbb{Q}[p_2, q_2, u][[z]]$ in which the sum of the exponents of p_2 and q_2 never exceeds the exponent of z to series of $\mathbb{Q}[s^{1/6}, s^{-1/6}][[t, \nu_\bullet, \nu_\circ]]$.

Note that (4) implies that the series $B \in \mathbb{Q}[p_2, q_2, u][[z]]$ satisfies the above condition.

Proof. Let us first observe that the series B defined by (5) is also the exponential generating function of bipartite maps (with vertex degrees 2 and 3 as before) *labelled on half-edges*, with half-edges weighted by $\sqrt{z}$. That is,

$$B = \sum_{\mathfrak{m}} \frac{z^{\mathrm{e}(\mathfrak{m})}}{\mathrm{e}(\mathfrak{m})!} u^{\mathrm{f}(\mathfrak{m})} \prod_{i=2}^{3} p_i^{\mathrm{v}_i^{\circ}(\mathfrak{m})} q_i^{\mathrm{v}_i^{\bullet}(\mathfrak{m})} = \sum_{\mathfrak{m}'} \frac{\sqrt{z}^{\mathrm{h}(\mathfrak{m}')}}{\mathrm{h}(\mathfrak{m}')!} u^{\mathrm{f}(\mathfrak{m}')} \prod_{i=2}^{3} p_i^{\mathrm{v}_i^{\circ}(\mathfrak{m}')} q_i^{\mathrm{v}_i^{\bullet}(\mathfrak{m}')},$$

where the first sum runs over edge labelled maps $\mathfrak{m}$, and the second over half-edge labelled maps $\mathfrak{m}'$. The notation $\mathrm{h}(\mathfrak{m}')$ stands for the number of half-edges of $\mathfrak{m}'$. The reason for that is that there exists a $(2n)!/n!$-to-1 correspondence between maps $\mathfrak{m}'$ with n edges labelled on half-edges and maps $\mathfrak{m}$ on n edges labelled on edges. This correspondence works as follows: starting from $\mathfrak{m}'$, we erase all labels that are incident to white vertices, and relabel the other half-edges with $1, 2, \ldots, n$, while preserving their relative order. This gives $\mathfrak{m}$. Conversely, starting from $\mathfrak{m}$, we first choose in $\{1, \ldots, 2n\}$ the n labels of $\mathfrak{m}'$ that will be incident to black vertices, spread them on the corresponding half-edges (while preserving the order of labels of $\mathfrak{m}$) and then distribute the remaining n labels in any way on the half-edges that are incident to white vertices. We obtain in this way $\binom{2n}{n} n! = (2n)!/n!$ maps $\mathfrak{m}'$. This observation will allow us to use the arguments of the theory of species [2] where now the atoms are half-edges (rather than edges in the proof of Proposition 2.2).

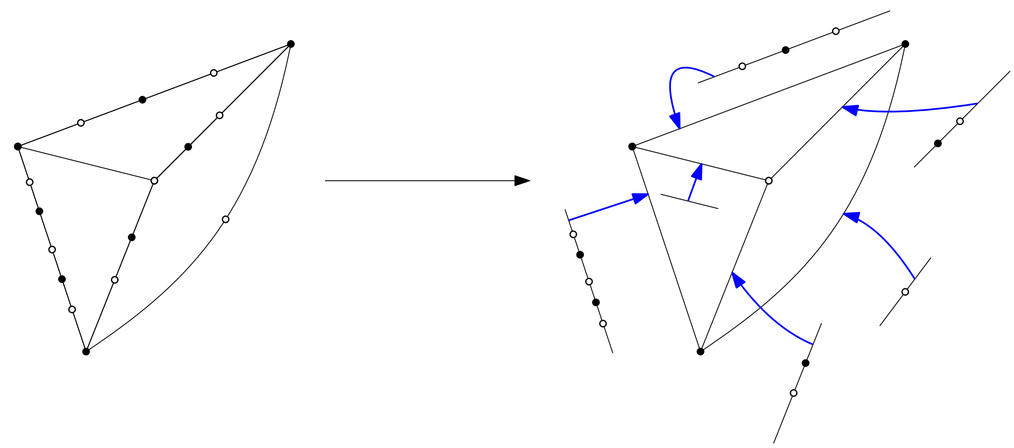

FIGURE 4. The decomposition of a bipartite map of $\mathcal{B}$ into a cubic map and chains. We refer to Figure 5 for details on the labels.

We now embark on the proof of (14). The maps $\mathfrak{m}'$ (labelled on half-edges) enumerated by B are of two types:

- Either they only have vertices of degree 2, in which case they are cycles of even length, say $2n$, enumerated by

$$2u^2 \sum_{n \geq 1} \frac{z^{2n}}{(4n)!} (p_2 q_2)^n (4n-1)! = \frac{u^2}{2} \log\left(\frac{1}{1 - z^2 p_2 q_2}\right).$$

  This can be seen by cutting these cycles in the middle of the edge containing the label 1, which gives an ordered chain with two dangling half-edges; the factor 2 accounts for the fact that the label 1 can be attached to a black or to a white vertex.
- Or they have vertices of degree 3, joined by chains of vertices of degree two (Figure 4, left). To such a map $\mathfrak{m}'$, we associate an Ising cubic map $\mathfrak{m}$ as follows: we erase all

vertices of degree 2, and only retain the labels that are incident to cubic vertices (Figure 5). If $\mathfrak{m}$ (and $\mathfrak{m}'$) have $2n$ cubic vertices, we then relabel these half-edges with labels $1,2,\dots,6n$, while preserving their order. Some oriented chains with dangling half-edges come out, as illustrated in Figures 4 and 5.

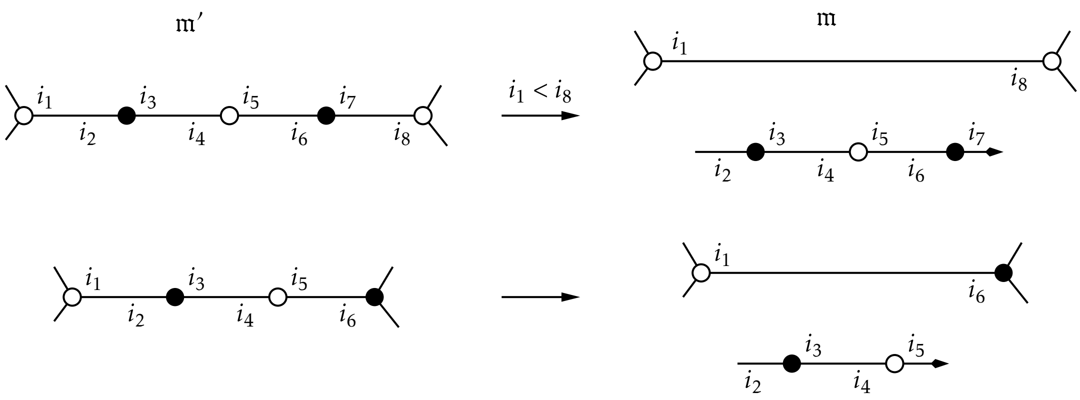

FIGURE 5. Erasing vertices of degree 2 in a bipartite map $\mathfrak{m}'$ gives an Ising cubic map $\mathfrak{m}$. Oriented bicoloured chains come out.

Let us determine the generating function of bipartite maps $\mathfrak{m}'$ that yield a given Ising cubic map $\mathfrak{m}$ having $2n$ vertices. These maps are obtained from $\mathfrak{m}$ by choosing for each edge e of $\mathfrak{m}$ a bipartite oriented chain with dangling half-edges at both ends (Figure 5):

- For a monochromatic white edge e with labels $i < j$, we choose a chain of $2k+1$ vertices (and $4k+2$ half-edges), with black endpoints. These chains are counted by
$$\sum_{k\geq 0} p_2^k q_2^{k+1} \sqrt{z}^{4k+2} = \frac{zq_2}{1-z^2p_2q_2}.$$
The starting point of the chain will be attached on the side of e labelled i.
- Analogously, for a monochromatic black edge e, we choose an oriented chain of $2k+1$ vertices, with white endpoints, with generating function $\frac{zp_2}{1-z^2p_2q_2}$; it will be inserted in e in a canonical fashion again.
- Finally, for a bicoloured edge e, we choose a (possibly empty) chain of $2k$ vertices, starting with a black vertex, with generating function $\frac{1}{1-z^2p_2q_2}$. This chain will be inserted in e in the only way that preserves bipartiteness.

Hence, the set of bipartite maps $\mathfrak{m}'$ that give $\mathfrak{m}$ after erasing vertices of degree 2 is a labelled product, in the species setting, of $\mathfrak{m}$ and of a collection of oriented chains, and has exponential generating function
$$\frac{\sqrt{z}^{6n}}{(6n)!} u^{\mathrm{f}(\mathfrak{m})} \left(\frac{zq_2}{1-z^2p_2q_2}\right)^{\mathrm{e}^\circ(\mathfrak{m})} \left(\frac{zp_2}{1-z^2p_2q_2}\right)^{\mathrm{e}^\bullet(\mathfrak{m})} \frac{1}{(1-z^2p_2q_2)^{3n-\mathrm{e}^\circ(\mathfrak{m})-\mathrm{e}^\bullet(\mathfrak{m})}}.$$
Recalling that $\mathfrak{m}$ has $3n$ edges and $2n$ vertices, and that its genus $\mathrm{g}(\mathfrak{m})$ satisfies $2\,\mathrm{g}(\mathfrak{m}) = 2+n-\mathrm{f}(\mathfrak{m})$, this can be rewritten as
$$\frac{u^2}{(2\,\mathrm{e}(\mathfrak{m}))!} \left(\frac{zu^{1/3}}{1-z^2p_2q_2}\right)^{\mathrm{e}(\mathfrak{m})} u^{-2\,\mathrm{g}(\mathfrak{m})} (zq_2)^{\mathrm{e}^\circ(\mathfrak{m})} (zp_2)^{\mathrm{e}^\bullet(\mathfrak{m})}.$$

Comparing with (12) shows that bipartite maps of $\mathcal{B}$ that have at least one vertex of degree 3 have generating function $u^2\Psi(I)$, where Ψ is the change of variables (13).

The rest of the proof is a mere calculation. □

3.2. A PDE for Ising cubic maps

In this subsection, we use the change of variables Φ to convert the PDE (11) satisfied by the generating function B of bipartite maps into a PDE for the generating function of Ising cubic maps I.

Let us return to the PDE (11), which involves series in z, p_2, q_2 and u. We want to apply to this identity the change of variable Φ defined by (16), and to write the resulting identity in terms of the series I (that is, in terms of ΦB) and its partial derivatives with respect to $t, \nu_\bullet, \nu_\circ$ and s. Recall that we set $p_3 = q_3 = 1$.

Lemma 3.2. *Applying the change of variable Φ to the differential operators involved in (11) yields:*

$$\Phi \circ z\frac{\partial}{\partial z} = \left(\frac{t(1+\nu_\bullet\nu_\circ)}{1-\nu_\bullet\nu_\circ}\frac{\partial}{\partial t} + \nu_\bullet\frac{\partial}{\partial \nu_\bullet} + \nu_\circ\frac{\partial}{\partial \nu_\circ}\right)\circ\Phi,$$

$$\Phi \circ \frac{\partial}{\partial p_2} = s^{1/6}t\left(t\nu_\circ\frac{\partial}{\partial t} + (1-\nu_\bullet\nu_\circ)\frac{\partial}{\partial \nu_\bullet}\right)\circ\Phi,$$

$$\Phi \circ L = s^{1/3}\Lambda\circ\Phi,$$

where L is defined by (6) and Λ is the following linear operator:

$$\Lambda = 2t^2\left(t(\nu_\circ^2+\nu_\bullet)\frac{\partial}{\partial t} + \nu_\circ(1-\nu_\bullet\nu_\circ)\frac{\partial}{\partial \nu_\bullet} + (1-\nu_\bullet\nu_\circ)\frac{\partial}{\partial \nu_\circ}\right). \tag{17}$$

Proof. Let $A \equiv A(z, p_2, q_2, u)$ be a series in $\mathbb{Q}[p_2, q_2, u][[z]]$ such that in all monomials of A, the sum of the exponents of p_2 and q_2 never exceeds the exponent of z. Let $J \equiv J(t, \nu_\bullet, \nu_\circ, s) = \Phi A$. This means conversely that

$$A(z, p_2, q_2, u) = \Psi J(t, \nu_\bullet, \nu_\circ, s) = J(\psi(z, p_2, q_2, u)), \tag{18}$$

where $\psi = (\psi_1, \dots, \psi_4)$ is the following vectorial function:

$$\psi : (z, p_2, q_2, u) \mapsto \left(u^{1/3}\frac{z}{1-z^2p_2q_2}, zp_2, zq_2, u^{-2}\right).$$

We differentiate (18) with respect to z using the chain rule, and then apply Φ:

$$\Phi\frac{\partial}{\partial z}A(z, p_2, q_2, u) = \Phi\left(\frac{\partial\psi_1}{\partial z}\right)\times\frac{\partial J}{\partial t}(t, \nu_\bullet, \nu_\circ, s) + \cdots + \Phi\left(\frac{\partial\psi_4}{\partial z}\right)\times\frac{\partial J}{\partial s}(t, \nu_\bullet, \nu_\circ, s).$$

So what we need is the image by Φ of the Jacobian matrix of ψ. We compute it to be:

$$\Phi\begin{pmatrix} \frac{\partial\psi_1}{\partial z} & \frac{\partial\psi_1}{\partial p_2} & \frac{\partial\psi_1}{\partial q_2} & \frac{\partial\psi_1}{\partial u} \\ \frac{\partial\psi_2}{\partial z} & \frac{\partial\psi_2}{\partial p_2} & \frac{\partial\psi_2}{\partial q_2} & \frac{\partial\psi_2}{\partial u} \\ \frac{\partial\psi_3}{\partial z} & \frac{\partial\psi_3}{\partial p_2} & \frac{\partial\psi_3}{\partial q_2} & \frac{\partial\psi_3}{\partial u} \\ \frac{\partial\psi_4}{\partial z} & \frac{\partial\psi_4}{\partial p_2} & \frac{\partial\psi_4}{\partial q_2} & \frac{\partial\psi_4}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{1+\nu_\bullet\nu_\circ}{s^{1/6}(1-\nu_\bullet\nu_\circ)^2} & t^2s^{1/6}\nu_\circ & t^2s^{1/6}\nu_\bullet & \cdots \\ \frac{\nu_\bullet}{ts^{1/6}(1-\nu_\bullet\nu_\circ)} & ts^{1/6}(1-\nu_\bullet\nu_\circ) & 0 & \cdots \\ \frac{\nu_\circ}{ts^{1/6}(1-\nu_\bullet\nu_\circ)} & 0 & ts^{1/6}(1-\nu_\bullet\nu_\circ) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We ignore the last row and column because we never differentiate with respect to u (see Corollary 2.3) and ψ_4 only depends on u.

We claim that this yields the identities stated in the lemma. Let us examine for instance the first one:

$$\Phi\left(z\frac{\partial A}{\partial z}\right) = \Phi\left(z\frac{\partial \psi_1}{\partial z}\right)\times\frac{\partial J}{\partial t} + \Phi\left(z\frac{\partial \psi_2}{\partial z}\right)\times\frac{\partial J}{\partial \nu_\bullet} + \Phi\left(z\frac{\partial \psi_3}{\partial z}\right)\times\frac{\partial J}{\partial \nu_\circ},$$
$$= t\frac{1+\nu_\bullet\nu_\circ}{1-\nu_\bullet\nu_\circ}\times\frac{\partial J}{\partial t} + \nu_\bullet\times\frac{\partial J}{\partial \nu_\bullet} + \nu_\circ\times\frac{\partial J}{\partial \nu_\circ},$$

as stated in the lemma.

The other identities are proved in a similar fashion. □

We can now write a PDE for the Ising generating function. We find convenient to use the following notation : for any variable a,

$$\partial_a = \frac{\partial}{\partial a} \qquad \text{and} \qquad D_a = a\frac{\partial}{\partial a}.$$

Proposition 3.3. *The generating function I of Ising cubic maps, defined by (12), satisfies the following fourth order PDE in the variables $t, \nu_\bullet$ and $\nu_\circ$:*

$$\Omega I = \frac{s}{12}\Lambda^4 I + \frac{1}{2}(\Lambda^2 I)^2 + t\left(\nu_\circ + 2t^3\left(2\nu_\circ^4 + \nu_\bullet\nu_\circ^2 + 2\nu_\bullet^2 + 3\nu_\circ\right)\right)\Lambda^2 I + t^5 Q, \tag{19}$$

where Λ is defined by (17),

$$\Omega := \frac{1}{3}\left(D_t + D_{\nu_\circ} - D_{\nu_\bullet} - 1\right)\circ\Lambda - \left(t\nu_\circ D_t + t(1-\nu_\bullet\nu_\circ)\partial_{\nu_\bullet}\right)^2, \tag{20}$$

and

$$Q = 2\nu_\circ\left(2\nu_\circ^4 + \nu_\bullet\nu_\circ^2 + 2\nu_\bullet^2 + 3\nu_\circ\right) + \left(\nu_\circ^5 + 2\nu_\circ^2 + \nu_\bullet\right)s + 2\left(2\nu_\circ^4 + \nu_\bullet\nu_\circ^2 + 2\nu_\bullet^2 + 3\nu_\circ\right)^2 t^3$$
$$+ 2\left(16\nu_\circ^8 + 5\nu_\circ^6\nu_\bullet + 10\nu_\bullet^2\nu_\circ^4 + 16\nu_\bullet^3\nu_\circ^2 + 59\nu_\circ^5 + 16\nu_\bullet^4 + 54\nu_\bullet\nu_\circ^3 + 37\nu_\bullet^2\nu_\circ + 32\nu_\circ^2 + 11\nu_\bullet\right)t^3 s.$$

We will see in the next section that this PDE, combined with a degree condition and the fact that I is symmetric in $\nu_\bullet$ and $\nu_\circ$, characterizes I in $t\mathbb{Q}[\nu_\bullet,\nu_\circ,s][[t]]$. Clearly, the PDE itself is not symmetric in $\nu_\bullet$ and $\nu_\circ$. So far, our efforts to build another PDE that would be both symmetric and smaller have failed.

Proof. We apply the change of variables Φ, defined by (16), to Equation (11), using the identities of Lemma 3.2 and the connection (15) between ΦB and I.

The left-hand side of (11) (with $p_3 = 1$) gives

$$\frac{1}{3s^{2/3}}\left(D_t + D_{\nu_\circ} - D_{\nu_\bullet} - 1\right)\circ\Lambda I + \frac{1}{s^{2/3}}t^2\nu_\circ^2.$$

The first term on the right-hand side gives

$$\frac{1}{s^{2/3}}\left(t\nu_\circ D_t + t(1-\nu_\bullet\nu_\circ)\partial_{\nu_\bullet}\right)^2 I + \frac{1}{2s^{2/3}}t^2\nu_\circ^2.$$

The second one gives

$$\frac{s^{1/3}}{12}\Lambda^4 I +$$
$$2s^{1/3}t^8\left(16\nu_\circ^8 + 5\nu_\bullet\nu_\circ^6 + 10\nu_\bullet^2\nu_\circ^4 + 16\nu_\bullet^3\nu_\circ^2 + 59\nu_\circ^5 + 16\nu_\bullet^4 + 54\nu_\bullet\nu_\circ^3 + 37\nu_\bullet^2\nu_\circ + 32\nu_\circ^2 + 11\nu_\bullet\right).$$

The third one gives

$$\frac{1}{2s^{2/3}}(\Lambda^2 I)^2+\frac{2}{s^{2/3}}t^4\left(2\nu_\circ^4+\nu_\bullet\nu_\circ^2+2\nu_\bullet^2+3\nu_\circ\right)\Lambda^2 I+\frac{2}{s^{2/3}}t^8\left(2\nu_\circ^4+\nu_\bullet\nu_\circ^2+2\nu_\bullet^2+3\nu_\circ\right)^2.$$

The fourth one gives

$$\frac{\nu_\circ}{s^{2/3}}t\Lambda^2 I+\frac{2}{s^{2/3}}t^5\nu_\circ\left(2\nu_\circ^4+\nu_\bullet\nu_\circ^2+2\nu_\bullet^2+3\nu_\circ\right).$$

Finally, the fifth and last one gives

$$\frac{1}{2s^{2/3}}t^2\left(2\nu_\circ^5 st^3+4\nu_\circ^2 st^3+2\nu_\bullet st^3+\nu_\circ^2\right).$$

It remains to multiply by $s^{2/3}$ to obtain the announced PDE. Note that the first term on the right-hand side of (11) has been moved to the left-hand side, for reasons that we will explain later. □

4. Uniqueness and effective calculation of the Ising series

4.1. Uniqueness

The first objective of this section is to establish the following result, according to which the PDE that we have obtained for the series I, combined with two natural conditions, characterizes this series.

Proposition 4.1. *The PDE (19) satisfied by the Ising series I of cubic maps, defined by (12), characterizes I in the ring of series $J \equiv J(t,\nu_\bullet,\nu_\circ,s)$ of $t\mathbb{Q}[\nu_\bullet,\nu_\circ,s][[t]]$ satisfying the following two conditions:*

- *for each n, the total degree in $\nu_\bullet$ and $\nu_\circ$ of the coefficient of t^n is bounded by n,*
- *$J(t,\nu_\bullet,0,s)=J(t,0,\nu_\bullet,s)$.*

More precisely, the PDE determines the coefficient of t^n in the series I inductively in n.

Observe that we do not need to require J to be a series in t^3. The above two conditions are obviously satisfied by I, because in the contribution of any Ising cubic map having n edges, the total degree in $\nu_\bullet$ and $\nu_\circ$ is the number of monochromatic edges, hence bounded by n. The symmetry is obvious as well, and more generally $I(t,\nu_\bullet,\nu_\circ,s)=I(t,\nu_\circ,\nu_\bullet,s)$.

Before we embark on the proof, let us examine more closely both sides of our PDE. Let $J=\sum_{n\ge 1}t^n J_n$ be a series satisfying the conditions of the proposition, where J_n is a polynomial in $\nu_\bullet,\nu_\circ$ and s. Then $\Omega(t^n J_n)$ is of the form $t^{n+2}\Omega_n(J_n)$, where $\Omega_n(J_n)$ is independent of t. More precisely, Ω_n is the following linear differential operator:

$$\Omega_n := \frac{2}{3}\left(n+1+D_{\nu_\circ}-D_{\nu_\bullet}\right)\circ\left((\nu_\circ^2+\nu_\bullet)n+\nu_\circ(1-\nu_\bullet\nu_\circ)\partial_{\nu_\bullet}+(1-\nu_\bullet\nu_\circ)\partial_{\nu_\circ}\right)$$
$$-\left(\nu_\circ(n+1)+(1-\nu_\bullet\nu_\circ)\partial_{\nu_\bullet}\right)\circ\left(\nu_\circ n+(1-\nu_\bullet\nu_\circ)\partial_{\nu_\bullet}\right).$$

Moreover, if we replace I by J in the right-hand side of the PDE (19), and extract the coefficient of t^{n+2}, then this coefficient only depends of the polynomials J_k up to $k=n-3$ (this is why we moved the term corresponding to $\partial^2 B/\partial {p_2}^2$ from one side of the PDE to the other). Hence, if we know the polynomials $J_0=0,J_1,\dots,J_{n-3}$, we can try to determine J_n by solving an equation of the form $\Omega_n(J_n)=\mathrm{Pol}$, for some explicit polynomial Pol in $\nu_\bullet,\nu_\circ$ and s. Since we know that $J=I$ is a solution, it suffices to study the kernel of Ω_n.

Lemma 4.2. *For $n \geq 1$, the linear operator Ω_n, restricted to polynomials P in $\nu_\bullet$ and $\nu_\circ$ (with coefficients in $\mathbb{Q}[s]$) of total degree at most n that satisfy $P(\nu_\bullet, 0) = P(0, \nu_\bullet)$, has trivial kernel.*

Proof. If $\Omega_n(P) = 0$, with
$$P(\nu_\bullet, \nu_\circ) = \sum_{i+j \leq n} p_{i,j} \nu_\bullet^i \nu_\circ^j,$$
then
$$3\Omega_n(P) = \sum_{i+j \leq n} p_{i,j} \nu_\bullet^i \nu_\circ^j \Big(2(j-n)(i-j-n)\nu_\bullet - (i-n)(i+2j-n+3)\nu_\circ^2 \\ -2j(i-j-n)\nu_\circ^{-1} + 2i(2i+j-2n)\nu_\bullet^{-1}\nu_\circ - 3i(i-1)\nu_\bullet^{-2} \Big) = 0.$$
Extracting the coefficient of $\nu_\bullet^{i+1} \nu_\circ^j$ gives a relation between the coefficients of P that we may try to use to compute them by decreasing induction on i (the *right-to-left* recursion):
$$2(n-j)(n-i+j)p_{i,j} = (i+1-n)(i+2j-n)p_{i+1,j-2} + 2(j+1)(i-j-n)p_{i+1,j+1} \\ -2(i+2)(2i+3+j-2n)p_{i+2,j-1} + 3(i+3)(i+2)p_{i+3,j}. \quad (21)$$
However, observe that the left-hand side vanishes at the two extreme points $(i,j) = (0,n)$ and $(i,j) = (n,0)$. Alternatively, we may try to compute $p_{i,j}$ by decreasing induction on j: extracting the coefficient of $\nu_\bullet^i \nu_\circ^{j+2}$ gives the following *top-down* recursion:
$$(i-n)(i+2j-n+3)p_{i,j} = 2(j+2-n)(i-3-j-n)p_{i-1,j+2} - 2(j+3)(i-3-j-n)p_{i,j+3} \\ +2(i+1)(2i+3+j-2n)p_{i+1,j+1} - 3(i+2)(i+1)p_{i+2,j+2}. \quad (22)$$
Of course, we have the boundary condition $p_{i,j} = 0$ if $i < 0$ or $j < 0$ or $i + j > n$.

We first write the top-down recursion (22) at $i = 0, j = n$, and obtain $p_{0,n} = 0$. By the symmetry assumption, we also have $p_{n,0} = 0$. Now for $i < n$ and $i+j \leq n$, the coefficient of $p_{i,j}$ in the right-to-left recursion (21) does not vanish, and allows us to conclude, by decreasing induction on i, that $p_{i,j} = 0$ for all i, so that the polynomial P is identically zero. ☐

Remark 4.3. The procedure used in the above proof also allows to solve the non-homogeneous equation $\Omega_n(J_n) = \mathrm{Pol}$ where Pol is the polynomial $\Omega_n(I_n)$, under the assumption that $J_n(\nu_\bullet, 0) = J_n(0, \nu_\bullet)$ and J_n has total degree at most n in $\nu_\bullet$ and $\nu_\circ$: we first determine the coefficient of $\nu_\bullet^0 \nu_\circ^n$ using the top-down recursion, then use symmetry to determine the coefficient of $\nu_\bullet^n \nu_\circ^0$, and then work by decreasing induction on the exponent of $\nu_\bullet$ using the right-left recursion.

Remark 4.4. If we do not impose the symmetry condition, the kernel of Ω_n appears to be trivial when n is even, but one-dimensional when n is odd. For instance, one readily checks that
$$\Omega_1(\nu_\bullet) = 0, \qquad \Omega_3(1 + \nu_\bullet^3) = 0, \qquad \Omega_5(\nu_\bullet^5 + 2\nu_\bullet^2 + \nu_\circ) = 0.$$

Proof of Proposition 4.1. Let us consider a series J satisfying the assumptions of the proposition. For $n \geq 0$, let J_n (resp. I_n) denote the coefficient of t^n in J (resp. I). Let us prove by induction on n that $J_n = I_n$. By assumption on J, this holds for $n = 0$. Assume that it holds for $J_0, J_1, \ldots, J_{n-1}$, with $n \geq 1$. Extract from the PDE (19) the coefficient of t^{n+2}. As observed above Lemma 4.2, this gives $\Omega_n J_n = \mathrm{Pol}$, where Pol only involves $\nu_\bullet, \nu_\circ, s$ and the polynomials $J_1, \ldots, J_{n-3}$ (and their partial derivatives). Since I also satisfies the PDE, and $I_i = J_i$ for $i < n$, we also have $\Omega_n I_n = \mathrm{Pol}$, *for the same value of* Pol. We conclude that $J_n = I_n$ thanks to Lemma 4.2. ☐

4.2. Implementation

We have implemented the above recursive calculation of the coefficient I_n of t^n in the series I both in Maple and in SageMath (see Remark 4.3). The codes are available on our webpages. We take advantage of the following three properties of I:

- $I_n = 0$ unless n is a multiple of 3,
- if we write $I_n = \sum_{i,j} I_{n,i,j} \nu_\bullet^i \nu_\circ^j$, for $I_{n,i,j}$ a polynomial in s, then $I_{n,i,j} = 0$ unless $i-j$ is a multiple of 3, as observed at the beginning of Section 3,
- I_n is symmetric in $\nu_\bullet$ and $\nu_\circ$.

With Maple, we reach $n = 72$ edges, with maximal genus 12, in a bit more than a minute. With SageMath, we get to a more modest $n = 54$ edges in 2-3 minutes, due to SageMath's less efficient handling of multivariate polynomials.

5. Three special cases

The form of the differential operators involved in the PDE satisfied by the Ising series I allows us to extract at once equations satisfied by three subseries of I: those counting planar maps, unicellular maps, and monochromatic (white) maps. In the latter case, we recover, unsurprisingly, the Goulden and Jackson recurrence relation on the number of cubic maps with $3n$ edges and genus g.

5.1. The planar case

This is the simplest possible specialization of the three. Let P be the Ising generating function of cubic planar maps, defined as in (12) but by restricting the sum to planar maps. It is obtained by setting $s = 0$ in I.

Corollary 5.1. *The Ising generating function P of cubic planar maps satisfies the following second order PDE in the variables $t, \nu_\bullet$ and $\nu_\circ$:*

$$\Omega P = \frac{1}{2}(\Lambda^2 P)^2 + t\left(\nu_\circ + 2t^3\left(2\nu_\circ^4 + \nu_\bullet\nu_\circ^2 + 2\nu_\bullet^2 + 3\nu_\circ\right)\right)\Lambda^2 P + t^5 Q_0,$$

where Λ is defined by (17), Ω by (20), and

$$Q_0 = 2\nu_\circ\left(2\nu_\circ^4 + \nu_\bullet\nu_\circ^2 + 2\nu_\bullet^2 + 3\nu_\circ\right) + 2\left(2\nu_\circ^4 + \nu_\bullet\nu_\circ^2 + 2\nu_\bullet^2 + 3\nu_\circ\right)^2 t^3.$$

Proof. This is obtained by setting $s = 0$ in the PDE (19) satisfied by I. A key property is that the operators Λ and Ω do not affect the exponent of s. □

Of course this is a complicated result, compared to the fact that the Ising generating function of *rooted* cubic maps, namely $6t\partial_t P$, is an explicit algebraic series in $t, \nu_\bullet, \nu_\circ$ (see [15, 4, 12]).

Remark 5.2. More generally, extracting from the PDE (19) the coefficient of s^g gives a PDE for the Ising generating function of maps of genus g in terms of the series counting maps of smaller genus.

5.2. The unicellular case

The planar case studied just above corresponds to maps having a maximal number of faces, given their edge number. Here we study the other extreme, with maps having a single face (also called *unicellular*), or equivalently maximal genus. Let U be the restriction to unicellular maps of the series I defined in (12). We furthermore set s to 1, as the edge number and the genus are then directly related by $\mathrm{e}(\mathfrak{m}) = 3(2\,\mathrm{g}(\mathfrak{m})-1)$.

Corollary 5.3. *The Ising generating function U of unicellular cubic maps satisfies the following fourth order* linear *PDE in the variables $t, \nu_\bullet$ and $\nu_\circ$:*

$$\Omega U = \frac{1}{12}\Lambda^4 U + t^5\left(\nu_\circ^5 + 2\nu_\circ^2 + \nu_\bullet\right),$$

where Λ is defined by (17) and Ω by (20).

Proof. The contribution of a coloured map $\mathfrak{m}$ in the Ising series I is of the form $t^{3n}s^g\nu_\circ^i\nu_\bullet^j$, and by Euler's relation, the number of faces of $\mathfrak{m}$ is then $2+n-2g$. Since there is at least one face, one always has $n \ge 2g-1$, and equality holds for unicellular maps only.

Let us now examine the effect of the operators Λ and Ω on a monomial $t^{\varepsilon+3(2g-1)}s^g\nu_\circ^i\nu_\bullet^j$, with $\varepsilon \ge 0$, and in particular on the exponents of t and s. We find:

$$\Lambda\left(t^{\varepsilon+3(2g-1)}s^g\nu_\circ^i\nu_\bullet^j\right) = t^{2+\varepsilon+3(2g-1)}s^g\lambda(\nu_\circ,\nu_\bullet),$$

$$\Omega\left(t^{\varepsilon+3(2g-1)}s^g\nu_\circ^i\nu_\bullet^j\right) = t^{2+\varepsilon+3(2g-1)}s^g\omega(\nu_\circ,\nu_\bullet),$$

for some functions λ and ω that do not involve t nor s. In particular,

$$s\Lambda^4\left(t^{\varepsilon+3(2g-1)}s^g\nu_\circ^i\nu_\bullet^j\right) = t^{8+\varepsilon+3(2g-1)}s^{g+1}\lambda_4(\nu_\circ,\nu_\bullet) = t^{2+\varepsilon+3(2(g+1)-1)}s^{g+1}\lambda_4(\nu_\circ,\nu_\bullet).$$

Hence, if we extract from (19) the monomials of the form $t^{2+3(2g-1)}s^g\nu_\circ^i\nu_\bullet^j$, we find that the terms involving $\Lambda^2 I$ do not contribute, that ΩI contributes ΩU, while $s\Lambda^4 I$ contributes $s\Lambda^4 U$. This gives the announced equation on U. □

Remark 5.4. We can also specialize the algorithm of Section 4.2 to compute inductively the coefficient of t^{3n} in the unicellular series U. In about one minute on a laptop one obtains for instance the Ising polynomials of unicellular cubic maps with at most 201 edges, which have genus at most 34.

5.3. The monochromatic white case, and the Goulden-Jackson recursion

We finally consider monochromatic white cubic maps, that is, those in which all edges are monochromatic white. Let M be the restriction to such maps of the series I defined by (12). We furthermore set $\nu_\circ = 1$ in this series, as this variable becomes redundant. Hence M is a series in t and s only.

Corollary 5.5. *The generating function M of cubic (uncoloured) maps, counted by edges (t) and genus (s) satisfies*

$$24t^7\left(tM''+3M'\right)^2 + 4s\,t^9M^{(4)} + 72s\,t^8M^{(3)} + t\left(348s\,t^6 + 48t^6 + 12t^3 - 1\right)M''$$
$$+ 4\left(105s\,t^6 + 36t^6 + 9t^3 - 1\right)M' + 3t^2\left(32s\,t^3 + 8t^3 + s + 4\right) = 0.$$

Equivalently, the number $r(n,g)$ of rooted cubic maps with $3n$ edges and genus g satisfies, for $n \geq 1$ and $g \geq 0$,

$$(n+1)r_{n,g} = 4n(3n-2)(3n-4)r_{n-2,g-1} + 4\sum_{i+j=n-2,\ h+k=g}(3i+2)(3j+2)r_{i,h}r_{j,k},$$

with the initial condition $r_{n,g} = \delta_{n,0}\delta_{g,0} - \frac{1}{2}\delta_{n,-1}\delta_{g,0}$ for $n \leq 0$ or $g < 0$.

The recursion was first established in [13]. Given the initial condition, the index i ranges from -1 to $n-1$ in the summation. The above differential can be rewritten in a more compact way upon introducing the series $R := 2t^3M' - 1/(2t) + t^2$:

$$R = 12t^6(R')^2 + 4st^9R''' + 36st^8R'' + t\left(60st^6 - 1\right)R'.$$

Proof. The contribution of a coloured map $\mathfrak{m}$ in the Ising series I is of the form $t^e\nu_\circ^i\nu_\bullet^j s^g$, with $e \geq i$. Equality means that the map is white monochromatic, and in this case $j = 0$.

Let us now examine the effect of the operators Λ and Ω on a monomial $m := t^{\varepsilon+i}\nu_\circ^i\nu_\bullet^j s^g$, with $\varepsilon \geq 0$, and in particular on the exponents of t and $\nu_\circ$. We will see that these operators do not decrease the difference between the exponent of t and the exponent of $\nu_\circ$. Define the following two operators:

$$\Lambda_\circ := 2t^2\nu_\circ^2(D_t - D_\bullet), \qquad \Omega_\circ := \frac{1}{3}(D_t + D_\circ - D_\bullet - 1)\circ\Lambda_\circ - (t\nu_\circ(D_t - D_\bullet))^2,$$

and write

$$\Lambda = \Lambda_\circ + \Lambda_1, \qquad \Omega = \Omega_\circ + \Omega_1.$$

Then

$$\Lambda_\circ(m) = \Lambda_\circ\left(t^{\varepsilon+i}\nu_\circ^i\nu_\bullet^j s^g\right) = t^{2+i+\varepsilon}\nu_\circ^{2+i}\lambda(\nu_\bullet, s),$$

$$\Omega_\circ(m) = \Omega_\circ\left(t^{\varepsilon+i}\nu_\circ^i\nu_\bullet^j s^g\right) = t^{2+i+\varepsilon}\nu_\circ^{2+i}\omega(\nu_\bullet, s),$$

for some functions λ and ω that do not involve t nor $\nu_\circ$. On the other hand, in all monomials occurring in $\Lambda_1(m)$ and $\Omega_1(m)$, the exponent of t exceeds the exponent of $\nu_\circ$ by at least $1+\varepsilon$.

Hence, if we extract from (19) the monomials where t and $\nu_\circ$ have the same exponent, we obtain

$$\Omega_\circ M_\circ = \frac{s}{12}\Lambda_\circ^4 M_\circ + \frac{1}{2}\left(\Lambda_\circ^2 M_\circ\right)^2 + t\nu_\circ(1+4t^3\nu_\circ^3)\Lambda_\circ^2 M_\circ + t^5\nu_\circ^5(4+s) + 8t^8\nu_\circ^8(1+4s),$$

where $M_\circ$ is obtained from I by extracting monomials where t and $\nu_\circ$ have the same power. Equivalently, $M_\circ$ is the series M evaluated at $t\nu_\circ$. This series does not involve the variable $\nu_\bullet$: hence, in the above identity, we can replace $\Lambda_\circ$ and $\Omega_\circ$, respectively, by

$$2t^2\nu_\circ^2 D_t \qquad \text{and} \qquad \frac{1}{3}(D_t + D_\circ - 1)\circ(2t^2\nu_\circ^2 D_t) - (t\nu_\circ D_t)^2.$$

Observe further that the operators D_t and $D_\circ$ act in the same way on monomials in which t and $\nu_\circ$ have the same exponent. Hence we can replace $D_\circ$ by D_t above. Setting finally $\nu_\circ = 1$ gives

$$\overline{\Omega}_\circ M = \frac{s}{12}\overline{\Lambda}_\circ^4 M + \frac{1}{2}(\overline{\Lambda}_\circ^2 M)^2 + t(1+4t^3)\overline{\Lambda}_\circ^2 M + t^5(4+s) + 8t^8(1+4s),$$

with

$$\overline{\Lambda}_\circ := 2t^2 D_t \qquad \text{and} \qquad \overline{\Omega}_\circ := \frac{1}{3}(2D_t - 1)\circ(2t^2 D_t) - (tD_t)^2. \tag{23}$$

This is the fourth order differential equation announced in the corollary. The recursion is obtained by writing

$$M = \sum_{n\ge 1, g\ge 0} \frac{r_{n,g}}{6n} t^{3n} s^g,$$

and extracting the coefficient of $t^{3n-1}s^g$ in the equation. □

Remark 5.6. Goulden and Jackson also derived the above recursion from the first KP equation (3). But they obtained it by forbidding, in bipartite maps, black vertices of degree 2 ($p_2 = 0$), while we obtain it, essentially, by forbidding black vertices of degree 3.

Remark 5.7. One can combine specializations. For instance, a DE for the generating function M_0 of planar white cubic maps is obtained by setting $s = 0$ in the DE of Corollary 5.5:

$$24t^7 (tM_0'' + 3M_0')^2 + t\left(48t^6 + 12t^3 - 1\right)M_0'' + 4\left(36t^6 + 9t^3 - 1\right)M_0' + 12t^2\left(2t^3 + 1\right) = 0.$$

Its solution, given by

$$2tM_0' = \sum_{n\ge 1} \frac{2\cdot 8^n}{(n+1)(n+2)} \binom{3n/2}{n} t^{3n},$$

has been known since the early work of Mullin, Nemeth and Schellenberg [20]. The above series satisfies a polynomial equation of degree 3.

At the other end of the genus scale, we can derive a linear DE for the generating function $U_\circ$ of unicellular white cubic maps by extracting from Corollary 5.3 the top contribution in terms of genus, as we did in the proof of Corollary 5.5. One obtains

$$\overline{\Omega}_\circ U_\circ = \frac{1}{12}\overline{\Lambda}_\circ^4 U_\circ + t^5,$$

with $\Omega_\circ$ and $\Lambda_\circ$ given by (23). That is,

$$4\left(105t^6 - 1\right)U_\circ' + t\left(348t^6 - 1\right)U_\circ'' + 72t^8 U_\circ^{(3)} + 4t^9 U_\circ^{(4)} + 3t^2 = 0.$$

Solving the associated recurrence relation on the coefficients of $U_\circ$ gives

$$U_\circ = \sum_{g\ge 1} \frac{(6g-4)!}{12^g g!(3g-2)!} t^{3(2g-1)}. \tag{24}$$

This result (or more precisely, a rooted version of it, that is, the series $2tU_\circ'$) can be found explicitly for instance in [8, Cor. 8], but is also equivalent to a special case of an older result due to Walsh and Lehman [26, Eq. 9]. In the next subsection, we return to the general unicellular case, and state a number of results and predictions on the value of other coefficients of the series U.

6. Inequalities

In this section, we prove Theorem 1.2.

Let us define an order relation between differential operators: given A and B two differential operators, we say that $A \ge B$ if for all nonnegative integers n, a, b with $3n \ge a + b$, for all reals x, y and all integers k

$$[t^k]\left((A-B)(t^{3n}\nu_\bullet^a \nu_\circ^b)\right)\Big|_{(\nu_\bullet,\nu_\circ)=(x,y)} \ge 0.$$

Lemma 6.1. *For all nonnegative integers n, a, b with $3n \geq a+b$, and for all positive integers k, one has the following inequalities*

$$\Lambda^k \geq 2^k t^{2k} \min(\nu_\circ^2, \nu_\bullet)^k D_t^k \geq 0 \tag{25}$$

$$\Omega \leq \frac{4t^2}{3}\left(\nu_\circ^2 + \nu_\bullet + \frac{\nu_\circ}{\nu_\bullet} + \frac{1}{\nu_\circ}\right)(D_t + 2) \circ D_t \tag{26}$$

Proof. The main fact that we use is that for n, a, b with $3n \geq a+b$, one has

$$(D_t - D_\circ - D_\bullet)(t^{3n}\nu_\bullet^a \nu_\circ^b) \geq 0$$

We first prove (25) for $k = 1$, for $k > 1$ it follows relatively easily by induction. Recall that by (17):

$$\begin{aligned}\Lambda &= 2t^2\left(t(\nu_\circ^2 + \nu_\bullet)\frac{\partial}{\partial t} + \nu_\circ(1 - \nu_\bullet\nu_\circ)\frac{\partial}{\partial \nu_\bullet} + (1 - \nu_\bullet\nu_\circ)\frac{\partial}{\partial \nu_\circ}\right)\\ &= 2t^2\left(\nu_\circ^2(D_t - D_\bullet) + \nu_\bullet(D_t - D_\circ)\frac{\partial}{\partial t} + \nu_\circ\frac{\partial}{\partial \nu_\bullet} + \frac{\partial}{\partial \nu_\circ}\right)\\ &\geq 2t^2\min(\nu_\circ^2, \nu_\bullet)(2D_t - D_\bullet - D_\circ) \geq 2t^2\min(\nu_\circ^2, \nu_\bullet)D_t.\end{aligned}$$

Now, we turn to (26). Since $D_t \geq D_\circ$ and $D_t \geq D_\bullet$, one gets

$$\Lambda \leq 2t^2\left(\nu_\circ^2 + \nu_\bullet + \frac{\nu_\circ}{\nu_\bullet} + \frac{1}{\nu_\circ}\right)D_t.$$

It is direct that

$$D_t + D_{\nu_\circ} - D_{\nu_\bullet} - 1 \leq 2D_t,$$

and finally we have

$$\left(t\nu_\circ D_t + t(1 - \nu_\bullet\nu_\circ)\partial_{\nu_\bullet}\right)^2 \geq \left(t\nu_\circ(D_t - D_{\nu_\bullet})\right)^2 \geq 0.$$

Combining these three inequalities in (20) yields (26).

□

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Note that $T = 2D_t I$ (by the correspondence between labelled and rooted maps), and that all the monomials in both I and T are of the form $t^{3n}\nu_\bullet^a\nu_\circ^b$ with $3n \geq a+b$ (because $3n$ counts edges, and $a+b$ counts monochromatic edges). Hence we can apply Lemma 6.1 and obtain

$$\Lambda^k I \geq 2^{k-1} t^{2k} \min(\nu_\circ^2, \nu_\bullet)^k D_t^{k-1} T \geq 0 \tag{27}$$

$$\Omega I \leq \frac{2t^2}{3}\left(\nu_\circ^2 + \nu_\bullet + \frac{\nu_\circ}{\nu_\bullet} + \frac{1}{\nu_\circ}\right)(D_t + 2)T \tag{28}$$

Where the inequalities hold for $\nu_\bullet, \nu_\circ > 0$ coefficientwise in s and t.
We can then plug these inequalities inside (19)and we directly obtain

$$\frac{2}{3}\left(\nu_\circ^2 + \nu_\bullet + \frac{\nu_\circ}{\nu_\bullet} + \frac{1}{\nu_\circ}\right)(D_t + 2)T \geq t^6\min(\nu_\circ^2, \nu_\bullet)^4\left(\frac{s}{12}D_t^3 T + 2(D_t T))^2\right)$$

Now, Theorem 1.2 follows by extracting the coefficient of $t^{3n}s^g$ above, and setting for instance

$$C(\nu_\bullet, \nu_\circ) = 10 \frac{\min(\nu_\circ^2, \nu_\bullet)^4}{\nu_\circ^2 + \nu_\bullet + \frac{\nu_\circ}{\nu_\bullet} + \frac{1}{\nu_\circ}}$$

□

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